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A Type of Fractional Power Series

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, a type of fractional power series is studied. We can find the exact solution of this fractional power series by using some methods. In fact, our result is a generalization of classical calculus result. Moreover, some examples are provided to illustrate our result. In fact, our result is a generalization of traditional calculus result.**

Keywords: **Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, exact solution, fractional power series.**

I. INTRODUCTION

Fractional calculus is a field of mathematical analysis. It studies and applies the integral and derivative of arbitrary order. In recent years, fractional calculus has been widely used in physics, engineering, economics, and other fields [1-12]. However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [13-16]. Since Jumarie's modified R-L fractional derivative helps avoid non-zero fractional derivative of constant functions, it is easier to use this definition to associate fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions we study the following α-fractional power series:

$$\sum_{k=0}^{\infty}\frac{k!\cdot(pk^2+qk+r)}{\Gamma(k\alpha+1)}x^{k\alpha}\ ,$$

where $0<\alpha\leq 1$, $-1<\frac{1}{\Gamma(\alpha+1)}x^{\alpha}<1$ and p,q,r are real numbers. The exact solution of this fractional power series can be obtained by using some techniques. In addition, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. Moreover, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([18])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{0}D_x^{\alpha}\right)\left[x^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_0D_x^{\alpha}\right)[C] = 0. \tag{3}$$

In the following, we introduce the definition of fractional analytic function.

Definition 2.3 ([19])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** If $0 < \alpha \le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) (x - x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha} \right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha} \right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^{\alpha} \right)^{\otimes_\alpha k}. \end{aligned} \tag{7}$$

Definition 2.5 ([21])**:** Suppose that $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the k-th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha (-1)}$.

III. RESULTS AND EXAMPLES

In this section, we will find the exact solution of a type of fractional power series. In addition, we provide some examples to illustrate our result. At first, we need a lemma.

Lemma 3.1: *Let* $0 < \alpha \le 1$ *and* $-1 < \frac{1}{\Gamma(\alpha+1)} x^\alpha < 1$. *Then*

$$\sum_{k=0}^{\infty} \frac{k!}{\Gamma(k\alpha+1)} x^{k\alpha} = \left(1 - \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (-1)}, \tag{8}$$

$$\sum_{k=0}^{\infty} \frac{k! \cdot (k+1)}{\Gamma(k\alpha+1)} x^{k\alpha} = \left(1 - \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (-2)}, \tag{9}$$

and

$$\sum_{k=0}^{\infty} \frac{k! \cdot k(k+1)}{\Gamma(k\alpha+1)} x^{k\alpha} = 2 \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha \left(1 - \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (-3)}. \tag{10}$$

Proof
$$\sum_{k=0}^{\infty}\frac{k!}{\Gamma(k\alpha+1)}x^{k\alpha}$$
$$=\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k}$$
$$=\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-1)}.$$

And
$$\sum_{k=0}^{\infty}\frac{k!\cdot(k+1)}{\Gamma(k\alpha+1)}x^{k\alpha}$$
$$=\sum_{k=0}^{\infty}(k+1)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k}$$
$$=\sum_{k=0}^{\infty}\left({}_0D_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (k+1)}\right]$$
$$=\left({}_0D_x^{\alpha}\right)\left[\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (k+1)}\right]$$
$$=\left({}_0D_x^{\alpha}\right)\left[\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k}\right]$$
$$=\left({}_0D_x^{\alpha}\right)\left[\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-1)}\right]$$
$$=\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-2)}.$$

On the other hand,
$$\sum_{k=0}^{\infty}\frac{k!\cdot k(k+1)}{\Gamma(k\alpha+1)}x^{k\alpha}$$
$$=\sum_{k=0}^{\infty}k(k+1)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k}$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\sum_{k=1}^{\infty}k(k+1)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (k-1)}$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\sum_{k=1}^{\infty}\left({}_0D_x^{\alpha}\right)^2\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (k+1)}\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left({}_0D_x^{\alpha}\right)^2\left[\sum_{k=1}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (k+1)}\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left({}_0D_x^{\alpha}\right)^2\left[\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k}\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left({}_0D_x^{\alpha}\right)^2\left[\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-1)}\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left[\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-2)}\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left[2\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-3)}\right]$$
$$=2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_\alpha\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (-3)}.$$
Q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1$, $-1 < \frac{1}{\Gamma(\alpha+1)}x^{\alpha} < 1$, *and* p, q, r *be real numbers. Then*

$$\sum_{k=0}^{\infty}\frac{k!\cdot(pk^2+qk+r)}{\Gamma(k\alpha+1)}x^{k\alpha}$$

$$= 2p\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-3)} + (q-p)\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-2)} + (r+p-q)\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-1)}. \quad (11)$$

Proof By Lemma 3.1,

$$\sum_{k=0}^{\infty}\frac{k!\cdot(pk^2+qk+r)}{\Gamma(k\alpha+1)}x^{k\alpha}$$

$$= \sum_{k=0}^{\infty}(pk^2+qk+r)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}k}$$

$$= \sum_{k=0}^{\infty}[pk(k+1)+(q-p)(k+1)+(r+p-q)]\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}k}$$

$$= p\cdot\sum_{k=0}^{\infty}k(k+1)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}k} + (q-p)\cdot\sum_{k=0}^{\infty}(k+1)\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}k} + (r+p-q)\cdot\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}k}$$

$$= 2p\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-3)} + (q-p)\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-2)} + (r+p-q)\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-1)}.$$

Q.e.d.

Example 3.3: Let $0 < \alpha \leq 1$ and $-1 < \frac{1}{\Gamma(\alpha+1)}x^{\alpha} < 1$, then

$$\sum_{k=0}^{\infty}\frac{k!\cdot(k^2+2k+3)}{\Gamma(k\alpha+1)}x^{k\alpha}$$

$$= 2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-3)} + \left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-2)} + 2\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-1)}. \quad (12)$$

And

$$\sum_{k=0}^{\infty}\frac{k!\cdot(5k^2-4k+2)}{\Gamma(k\alpha+1)}x^{k\alpha}$$

$$= 10\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes_{\alpha}\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-3)} - 9\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-2)} + 11\cdot\left(1-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(-1)}. \quad (13)$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, a type of fractional power series is studied. We can find the exact solution of this fractional power series by using some techniques. In fact, our result is a generalization of traditional calculus result. In the future, we will continue to use our methods to study the problems in fractional differential equations and applied mathematics.

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